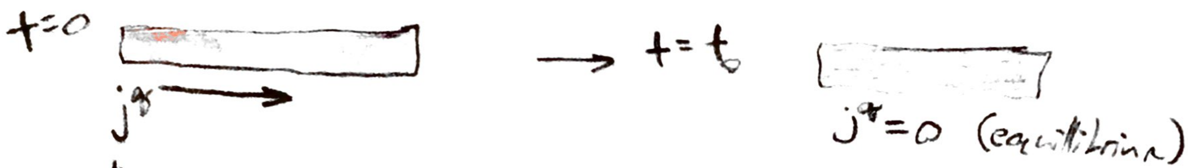
Thermal Conductivity using Drude Model

[Intro to j^{q} and κ]

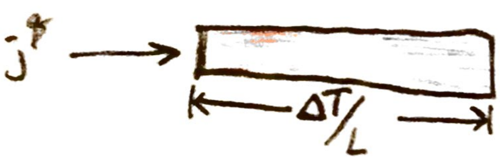
Assume that the bulk of thermal current in a metal is carried by e^-'s

Imagine a metal bar that is hot on one side and kool on the other at t=0 as such:



j^{q} will carry heat until the hot end and cool end both reach some midpoint of equillibrium.

However, if we supply a constant source of j^{q}, then we can maintain a constant gradient across the bar:



thus, $j^{q} \propto \Delta T/L$

or more formally, we can define κ s.t.:

$$\vec{j}^{q} = -\kappa \vec{\nabla} T$$

where the - sign is meant to establish the convention "heat flows from hot to cold" and $\kappa > 0$

note: this implies that κ is either a scalar or a tensor

[Electron thermal current]

$j^q \equiv$ rate of energy transfer per cross-sectional area

rate $\Rightarrow \vec{v}$ or τ

energy $\Rightarrow \Delta\mathcal{E}$

per area $\Rightarrow$ electron density $\Rightarrow n$

since we are *adding up* the contributions from individual e^-'s, we will be doomed to forget the $\frac{1}{2}$ somewhere...

$$j^q = \frac{1}{2} n v \underbrace{\left[\mathcal{E}(\text{hot end}) - \mathcal{E}(\text{cold end})\right]}_{= d\mathcal{E} = \frac{d\mathcal{E}}{dT} dT}$$

$$j^q = n \frac{d\mathcal{E}}{dt} \cdot v \cdot \underbrace{dT}_{= dx \frac{dT}{dx} \rightarrow v\tau \frac{dT}{dx}}$$

Finally, we can say in general:

from $\langle v^2 \rangle = 3\langle v_x^2 \rangle$

$$\vec{j}^q = -\left(\frac{1}{3} n v^2 \tau \frac{d\mathcal{E}}{dT}\right) \vec{\nabla} T = -K \vec{\nabla} T$$

$\frac{d\mathcal{E}}{dT} \equiv c_v / n$

$$\Rightarrow \boxed{K = \frac{1}{3} v^2 \tau C_v}$$

[Weiderman-Franz law]

now that we know that:

$$K = \frac{1}{3} v^2 \tau C_v$$

now recall that, from Drude:

$$\sigma = ne^2/m$$

it then follows that:

$$\frac{K}{\sigma} = \frac{\frac{1}{3} C_v m v^2}{ne^2}$$ ← potentially useful formula

Treating electrons as an ideal gas

$$\frac{1}{2} m v^2 \rightarrow \frac{3}{2} k_B T$$

$$C_v \rightarrow \frac{3}{2} n k_B$$

we can now obtain Weiderman-Franz law:

$$\boxed{\frac{K}{\sigma T} = \frac{3}{2}\left(\frac{k_B}{e}\right)^2 = \text{constant}}$$

$$\boxed{\text{Lorenz number} = \frac{3}{2}\left(\frac{k_B}{e}\right)^2 = 1.11 \times 10^{-8} \ \frac{W \cdot \Omega}{K^2}}$$