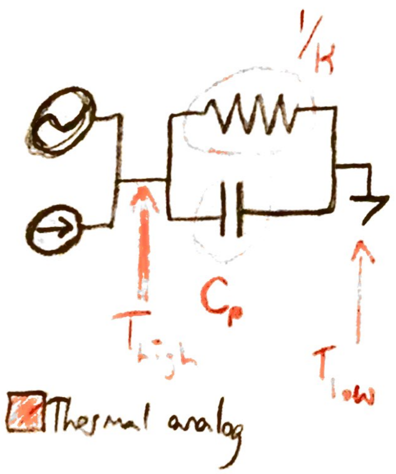
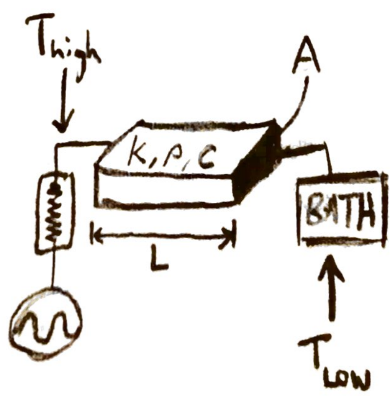
[Parallels between electrical and thermal transport]

Electrical	thermal
$\vec{j} = -\sigma \vec{\nabla} V$	$\vec{j}^{\,q} = -\kappa \vec{\nabla} T$
continuity: $\vec{\nabla} \cdot \vec{j} + \frac{\partial \rho}{\partial t} = 0$	heat equation: $\vec{\nabla} \cdot \vec{j}^{\,q} + \rho C_p \frac{\partial T}{\partial t} = 0$
Laplace: $\sigma \nabla^2 V = 0$	steady-state: $\kappa \nabla^2 T = 0$

[Comparing AC electric and AC thermal transport]





Start with heat equation:

$$\underbrace{-\kappa}_{\text{thermal conductivity}} \nabla^2 T + \underbrace{\rho}_{\text{density}} \underbrace{C_p}_{\text{heat capacity}} \frac{\partial T}{\partial t} = 0$$

Then integrate over volume:

$$\int\!\!\int \left[-\kappa \nabla^2 T + \rho C_p \frac{\partial T}{\partial t} \right] dV = 0$$

$\nabla \cdot \nabla T$

[AC thermal cont.]

the first term, by divergence theorem, is:

$$\int dV\, \nabla\cdot(-k\nabla T) = \oint ds\, (-k\nabla T)$$

by magic?:

$$\oint ds\,(-k\nabla T) = -k\vec{\nabla}T\times A = \ldots = \boxed{KT}$$

where $\boxed{K \equiv k\cdot A}$

the second term, since $\frac{\partial T}{\partial t}$ is assumed to be uniform:

$$\int c_p \rho \frac{\partial T}{\partial t}\, dV = \boxed{C_p \frac{\partial T}{\partial t}}$$

where $\boxed{C_p = c_p\cdot m}$ or $\int c_p \rho\, dV$

now:

$$\boxed{KT + C_p \frac{\partial T}{\partial t} = \dot{Q}_{source}}$$

for $\dot{Q}_{source}$ is heat supplied by heater

by joule heating, this is:

$$\dot{Q}_{source} = I^2 R = I_o^2 R_{heater} \sin^2(\omega t) = \frac{I_o^2 R_{heater}}{2}\left[1-\cos(2\omega t)\right]$$

$$\therefore\ \dot{Q}_{source} = \underbrace{\left(\frac{I_o^2 R_H}{2}\right)}_{\dot{Q}_{DC}} - \underbrace{\left(\frac{I_o^2 R_H}{2}\right)\cos(2\omega t)}_{\dot{Q}_{AC}}$$

then, we can solve each part independently due to the orthogonality of $\cos(\omega t)$

$$K T^{dc} + C_p \frac{\partial T^{dc}}{\partial t} = \dot{Q}^{dc}_{source}$$

$$K T^{ac} + C_p \frac{\partial T^{ac}}{\partial t} = \dot{Q}^{2\omega}_{source}$$

initial condition: $\dot{Q} = 0$, gives solution:

$$\boxed{T^{dc}(t) = T_{bath} + \frac{1}{2} T_0 \left[1 - \exp\left[-t/\tau_{q}\right]\right]}$$

where $\frac{1}{2} T_0 \equiv \frac{I_0 R^2}{2K}$ and $\tau_q \equiv \frac{C_p}{K}$

error in notes: eq'n 43: $C_p \to t$

For $\dot{Q} \neq 0$: Use Electrical AC w/ following replacements:

$V \to T$

$\omega \to 2\omega$

$I_0 \to -\frac{I^2 R_H}{2}$

$\frac{1}{R} \to K$

$C \to C_p$

then,

$$\boxed{R_e[T^{a.c}] = \frac{-T_0}{2\sqrt{1 + 4\omega^2 \tau_{q}^2}} \cos\left[2\omega t + \tan^{-1}(2\omega \tau_q)\right]}$$

and the total solution is:

$$\boxed{T = T^{dc} + T^{ac} = \underbrace{T_{bath} + \frac{1}{2} T_0 \left[1 - e^{-t/\tau_q}\right]}_{T^{dc}} + \underbrace{\frac{T_0}{2\sqrt{1 + 4\omega^2 \tau_q^2}} \cos\left[2\omega t + \tan^{-1}(2\omega \tau_q)\right]}_{T^{ac}}}$$

[Limiting Cases]

Long time ($t \gg \tau$):

$$T(t) \approx T_{bath} + \frac{T_0}{2}\left(1 + \frac{\cos\left[2\omega t + \tan^{-1}(2\omega\tau_c)\right]}{\sqrt{1+4\omega^2\tau^2}}\right)$$

dc limit ($\omega \to 0$):

$$T(t) \approx T_{bath} + T_0\left(1 - \frac{1}{2}e^{-t/\tau_c}\right)$$

or, as we know experimentally:

$$T_0 - T_{bath} = I_0^2 R/K$$

low-frequency limit: ($\omega\tau_c \ll 1$):

$$T(t) \to T_{bath} + T_0 \cos^2(\omega t)$$

high-frequency limit:

$$T(t) \longrightarrow T_{bath} + \frac{T_0}{2} + \frac{T_0}{2}\frac{\sin(2\omega t)}{2\omega\tau} \longrightarrow T_{bath} + \frac{T_0}{2}$$